Table of Contents

Introduction xv

Chapter 1. Scalar Hyperbolic Conservation Laws in One Dimension of Space 1

1.1. Definitions 1
1.1.1. Hyperbolic scalar conservation laws 1
1.1.2. Derivation from general conservation principles 3
1.1.3. Non-conservation form 6
1.1.4. Characteristic form – Riemann invariants 7
1.2. Determination of the solution 9
1.2.1. Representation in the phase space 9
1.2.2. Initial conditions, boundary conditions 12
1.3. A linear law: the advection equation 14
1.3.1. Physical context – conservation form 14
1.3.2. Characteristic form 16
1.3.3. Example: movement of a contaminant in a river 17
1.3.4. Summary 21
1.4. A convex law: the inviscid Burgers equation 21
1.4.1. Physical context – conservation form 21
1.4.2. Characteristic form 22
1.4.3. Example: propagation of a perturbation in a fluid 24
1.4.4. Summary 28
1.5. Another convex law: the kinematic wave for free-surface hydraulics 28
1.5.1. Physical context – conservation form 28
1.5.2. Non-conservation and characteristic forms 30
1.5.3. Expression of the wave speed. 31
1.5.4. Particular case: flow in a rectangular channel 34

1.5.5. Summary 35
1.6. A non-convex conservation law: the Buckley-Leverett equation 35
1.6.1. Physical context – conservation form 35
1.6.2. Characteristic form 38
1.6.3. Example: decontamination of an aquifer 40
1.6.4. Summary 41
1.7. Advection with adsorption/desorption 42
1.7.1. Physical context – conservation form 42
1.7.2. Characteristic form 45
1.7.3. Summary 47
1.8. Summary of Chapter 1 47
1.8.1. What you should remember 47
1.8.2. Application exercises 48

Chapter 2. Hyperbolic Systems of Conservation Laws in One Dimension of Space 53

2.1. Definitions 53
2.1.1. Hyperbolic systems of conservation laws. 53
2.1.2. Hyperbolic systems of conservation laws – examples 55
2.1.3. Characteristic form – Riemann invariants 57
2.2. Determination of the solution 59
2.2.1. Domain of influence, domain of dependence 59
2.2.2. Existence and uniqueness of solutions – initial and boundary conditions 61
2.3. A particular case: compressible flows 63
2.3.1. Definition 63
2.3.2. Conservation form 63
2.3.3. Characteristic form 66
2.3.4. Physical interpretation 67
2.4. A linear 2×2 system: the water hammer equations 68
2.4.1. Physical context – assumptions. 68
2.4.2. Conservation form 70
2.4.3. Characteristic form – Riemann invariants 75
2.4.4. Calculation of the solution 79
2.4.5. Summary 83
2.5. A nonlinear 2×2 system: the Saint Venant equations 84
2.5.1. Physical context – assumptions. 84
2.5.2. Conservation form 85
2.5.3. Characteristic form – Riemann invariants 91
2.5.4. Calculation of solutions 100
2.5.5. Summary 107

2.6. A nonlinear 3×3 system: the Euler equations 108
2.6.1. Physical context – assumptions. 108
2.6.2. Conservation form 109
2.6.3. Characteristic form – Riemann invariants 113
2.6.4. Calculation of the solution 117
2.6.5. Summary 121
2.7. Summary of Chapter 2 122
2.7.1. What you should remember 122
2.7.2. Application exercises 123

Chapter 3. Weak Solutions and their Properties 131

3.1. Appearance of discontinuous solutions 131
3.1.1. Governing mechanisms 131
3.1.2. Local invalidity of the characteristic formulation – graphical approach 134
3.1.3. Practical examples of discontinuous flows 136
3.2. Classification of waves 138
3.2.1. Shock wave 138
3.2.2. Rarefaction wave 140
3.2.3. Contact discontinuity. 140
3.2.4. Mixed/compound wave 141
3.3. Simple waves. 142
3.3.1. Definition and properties 142
3.3.2. Generalized Riemann invariants 143
3.4. Weak solutions and their properties 144
3.4.1. Definitions 144
3.4.2. Non-equivalence between the formulations 145
3.4.3. Jump relationships 146
3.4.4. Non-uniqueness of weak solutions. 148
3.4.5. The entropy condition 152
3.4.6. Irreversibility 154
3.4.7. Approximations for the jump relationships. 156
3.5. Summary 157
3.5.1. What you should remember 157
3.5.2. Application exercises 158

Chapter 4. The Riemann Problem 161

4.1. Definitions – solution properties 161
4.1.1. The Riemann problem 161
4.1.2. The generalized Riemann problem. 162
4.1.3. Solution properties 163

4.2. Solution for scalar conservation laws 165
4.2.1. The linear advection equation 165
4.2.2. The inviscid Burgers equation 166
4.2.3. The Buckley-Leverett equation 168
4.3. Solution for hyperbolic systems of conservation laws 173
4.3.1. General principle 173
4.3.2. Application to the water hammer problem: sudden valve failure . . 174
4.3.3. Free surface flow: the dambreak problem 177
4.3.4. The Euler equations: the shock tube problem 183
4.4. Summary 189
4.4.1. What you should remember 189
4.4.2. Application exercises 190

Chapter 5. Multidimensional Hyperbolic Systems 193

5.1. Definitions 193
5.1.1. Scalar laws 193
5.1.2. Two-dimensional hyperbolic systems 195
5.1.3. Three-dimensional hyperbolic systems 196
5.2. Derivation from conservation principles 197
5.3. Solution properties 200
5.3.1. Two-dimensional hyperbolic systems 200
5.3.2. Three-dimensional hyperbolic systems 206
5.4. Application: the two-dimensional shallow water equations 208
5.4.1. Governing equations 208
5.4.2. The secant plane approach 213
5.4.3. Interpretation – determination of the solution 218
5.5. Summary 221
5.5.1. What you should remember 221
5.5.2. Application exercises 221

Chapter 6. Finite Difference Methods for Hyperbolic Systems 223

6.1. Discretization of time and space 223
6.1.1. Discretization for one-dimensional problems 223
6.1.2. Multidimensional discretization 224
6.1.3. Explicit schemes, implicit schemes 226
6.2. The method of characteristics (MOC) 227
6.2.1. MOC for scalar hyperbolic laws 227
6.2.2. The MOC for hyperbolic systems of conservation laws 235
6.2.3. Application examples 240
6.3. Upwind schemes for scalar laws 244
6.3.1. The explicit upwind scheme (non-conservative version) 244
6.3.2. The implicit upwind scheme (non-conservative version) 245

6.3.3. Conservative versions of the implicit upwind scheme 247
6.3.4. Application examples 249
6.4. The Preissmann scheme 250
6.4.1. Formulation 250
6.4.2. Estimation of nonlinear terms – algorithmic aspects 253
6.4.3. Numerical applications 254
6.5. Centered schemes 260
6.5.1. The Crank-Nicholson scheme 260
6.5.2. Centered schemes with Runge-Kutta time stepping. 261
6.6. TVD schemes 263
6.6.1. Definitions 263
6.6.2. General formulation of TVD schemes. 264
6.6.3. Harten's and Sweby's criteria 266
6.6.4. Classical limiters 268
6.6.5. Computational example 269
6.7. The flux splitting technique 271
6.7.1. Principle of the approach 271
6.7.2. Application to classical schemes 274
6.8. Conservative discretizations: Roe's matrix 280
6.8.1. Rationale and principle of the approach. 280
6.8.2. Expression of Roe's matrix 281
6.9. Multidimensional problems 284
6.9.1. Explicit alternate directions 284
6.9.2. The ADI method 286
6.9.3. Multidimensional schemes 288
6.10. Summary 289
6.10.1. What you should remember 289
6.10.2. Application exercises 291

Chapter 7. Finite Volume Methods for Hyperbolic Systems 293

7.1. Principle 293
7.1.1. One-dimensional conservation laws 293
7.1.2. Multidimensional conservation laws 295
7.1.3. Application to the two-dimensional shallow water equations 297
7.2. Godunov's scheme 299
7.2.1. Principle 299
7.2.2. Application to the scalar advection equation 301
7.2.3. Application to the inviscid Burgers equation. 305
7.2.4. Application to the water hammer equations 308
7.3. Higher-order Godunov-type schemes 313
7.3.1. Rationale and principle 313
7.3.2. Example: the MUSCL scheme 316

7.4. EVR approach 319
7.4.1. Principle of the approach 319
7.4.2. Application to the one-dimensional shallow water equations 323
7.5. Summary 326
7.5.1. What you should remember 326
7.5.2. Application exercises 327

Chapter 8. Finite Element Methods for Hyperbolic Systems 329

8.1. Principle for one-dimensional scalar laws 329
8.1.1. Weak form 329
8.1.2. Discretization of space and time 330
8.1.3. Classical shape and test functions 335
8.2. One-dimensional hyperbolic systems 340
8.2.1. Weak formulation. 340
8.2.2. Application to the non-conservation form 341
8.3. Extension to multidimensional problems 344
8.3.1. Weak form of the equations. 344
8.3.2. Discretization of space 345
8.3.3. Classical shape and test functions 345
8.4. Discontinuous Galerkin techniques. 347
8.4.1. Principle of the method 347
8.4.2. Legendre polynomial-based reconstruction 349
8.4.3. Limiting 351
8.4.4. Runge-Kutta time stepping 353
8.5. Application examples 354
8.5.1. The linear advection equation 354
8.5.2. The inviscid Burgers equation 359
8.6. Summary 368
8.6.1. What you should remember 368
8.6.2. Application exercises 369

Chapter 9. Treatment of Source Terms. 371

9.1. Introduction. 371
9.2. Problem position 372
9.2.1. Example 1: the water hammer equations 372
9.2.2. Example 2: the shallow water equations 374
9.2.3. Stationary solution and C–property 376
9.3. Source term upwinding techniques 377
9.3.1. Principle 377
9.3.2. Application example 1: the water hammer equations. 380
9.3.3. Application example 2: the shallow water equations with HLL solver 382

9.4. The quasi-steady wave algorithm 386
9.4.1. Principle 386
9.4.2. Application to the water hammer equations 387
9.4.3. Application to the one-dimensional shallow water equations 387
9.5. Balancing techniques 390
9.5.1. Well-balancing 390
9.5.2. Hydrostatic pressure reconstruction for free surface flow 393
9.5.3. Auxiliary variable-based balancing 395
9.6. Computational example 403
9.7. Summary 408

Chapter 10. Sensitivity Equations for Hyperbolic Systems 411

10.1. Introduction 411
10.2. Forward sensitivity equations for scalar laws 413
10.2.1. Derivation for continuous solutions 413
10.2.2. Conservation, non-conservation and characteristic forms 415
10.2.3. Extension to discontinuous solutions 416
10.2.4. Solution of the Riemann problem 418
10.3. Forward sensitivity equations for hyperbolic systems 422
10.3.1. Governing equations 422
10.3.2. Non-conservation and characteristic forms 424
10.3.3. The Riemann problem 426
10.3.4. Application example: the one-dimensional shallow water sensitivity equations 427
10.4. Adjoint sensitivity equations 435
10.4.1. Introduction 435
10.4.2. Adjoint models for scalar laws 435
10.5. Finite volume solution of the forward sensitivity equations 441
10.5.1. Introduction 441
10.5.2. Discretization 442
10.5.3. A modified HLL Riemann solver for sensitivity solutions 443
10.5.4. Application example: the one-dimensional shallow water equations 446
10.6. Summary 447

Chapter 11. Modeling in Practice 449

11.1. Modeling software 449
11.1.1. Introduction 449
11.1.2. Conservation 450
11.1.3. Solution monotony 453
11.2. Mesh quality 454
11.3. Boundary conditions 459

11.3.1. Number and nature of boundary conditions . . . 459
11.3.2. Prescribed discharge/flow velocity . . . 460
11.3.3. Prescribed pressure/water level . . . 461
11.3.4. Stage-discharge and pressure-discharge relationships . . . 463
11.4. Numerical parameters . . . 464
11.4.1. Computational time step . . . 464
11.4.2. Scheme centering parameters . . . 465
11.4.3. Iteration control . . . 465
11.5. Simplifications in the governing equations . . . 466
11.5.1. Rationale . . . 466
11.5.2. The Local Partial Inertia (LPI) technique . . . 467
11.5.3. The Reduced Momentum Equation (RME) technique . . . 468
11.5.4. Application examples . . . 469
11.6. Numerical solution assessment . . . 472
11.6.1. Software solution accuracy . . . 472
11.6.2. Assessing solution convergence . . . 473
11.6.3. Consistency analysis – numerical diffusion and dispersion . . . 474
11.6.4. Stability analysis – phase and amplitude portraits . . . 476
11.7. Getting started with a simulation package . . . 477

Appendix A. Linear Algebra . . . 479

A.1. Definitions . . . 479
A.2. Operations on matrices and vectors . . . 480
A.2.1. Addition . . . 480
A.2.2. Multiplication by a scalar . . . 481
A.2.3. Matrix product . . . 481
A.2.4. Determinant of a matrix . . . 482
A.2.5. Inverse of a matrix . . . 482
A.3. Differential operations using matrices and vectors . . . 483
A.3.1. Differentiation . . . 483
A.3.2. Jacobian matrix . . . 483
A.4. Eigenvalues, eigenvectors . . . 483
A.4.1. Definitions . . . 483
A.4.2. Example . . . 484

Appendix B. Numerical Analysis . . . 487

B.1. Consistency . . . 487
B.1.1. Definitions . . . 487
B.1.2. Principle of a consistency analysis . . . 487
B.1.3. Numerical diffusion, numerical dispersion . . . 489
B.2. Stability . . . 491
B.2.1. Definition . . . 491

B.2.2. Principle of a stability analysis . . . 492
B.2.3. Harmonic analysis of analytical solutions . . . 494
B.2.4. Harmonic analysis of numerical solutions . . . 497
B.2.5. Amplitude and phase portraits . . . 499
B.2.6. Extension to systems of equations . . . 501
B.3. Convergence . . . 503
B.3.1. Definition . . . 503
B.3.2. Lax's theorem . . . 503

Appendix C. Approximate Riemann Solvers . . . 505

C.1. The HLL and HLLC solvers . . . 505
C.1.1. The HLL solver . . . 505
C.1.2. The HLLC solver . . . 508
C.2. Roe's solver . . . 511
C.2.1. Principle . . . 511
C.2.2. Algorithmic simplification . . . 513
C.2.3. Entropy violation and fixes . . . 514
C.2.4. Application example: the shallow water equations . . . 514
C.3. The Lax-Friedrichs solver . . . 515
C.4. Approximate-state solvers . . . 516
C.4.1. Principle . . . 516
C.4.2. Shock-based solvers . . . 516
C.4.3. Rarefaction wave-based solvers . . . 517

Appendix D. Summary of the Formulae . . . 521

Bibliography . . . 527

Index . . . 537