

## Table of Contents

<b>Preface</b> . . . . .	xi
<b>Chapter 1. Probability and Random Variables</b> . . . . .	1
1.1. Introductory notes . . . . .	1
1.2. Probability space . . . . .	2
1.3. Conditional probability and independence . . . . .	8
1.4. Random variables . . . . .	12
1.4.1. Discrete random variables . . . . .	14
1.4.2. Bernoulli random variables . . . . .	15
1.4.3. Binomial random variables . . . . .	15
1.4.4. Geometric random variables . . . . .	16
1.4.5. Poisson random variables . . . . .	17
1.4.6. Continuous random variables . . . . .	18
1.4.7. Exponential random variables . . . . .	20
1.4.8. Uniform random variables . . . . .	21
1.4.9. Gamma random variables . . . . .	21
1.4.10. Normal random variables . . . . .	22
1.4.11. Lognormal random variables . . . . .	23
1.4.12. Weibull random variables . . . . .	23
1.5. Expectation and variance of a random variable . . . . .	24
1.6. Jointly distributed random variables . . . . .	28
1.6.1. Joint probability distribution of functions of random variables . . . . .	30
1.7. Moment generating functions . . . . .	32
1.8. Probability inequalities and limit theorems . . . . .	37
1.9. Multivariate normal distribution . . . . .	44
<b>Chapter 2. An Introduction to Financial Instruments and Derivatives</b> . . . . .	49
2.1. Introduction . . . . .	49
2.2. Bonds and basic interest rates . . . . .	50

2.2.1. Simple interest rates . . . . .	51
2.2.2. Discretely compounded interest rates . . . . .	51
2.2.3. Continuously compounded interest rate . . . . .	52
2.2.4. Money-market account . . . . .	53
2.2.5. Basic interest rates . . . . .	55
2.2.5.1. Treasury rate . . . . .	55
2.2.5.2. LIBOR rates . . . . .	55
2.2.6. Time value of money . . . . .	55
2.2.7. Coupon-bearing bonds and yield-to-maturity . . . . .	56
2.3. Forward contracts . . . . .	58
2.3.1. Arbitrage . . . . .	59
2.4. Futures contracts . . . . .	60
2.5. Swaps . . . . .	60
2.6. Options . . . . .	62
2.6.1. European call option . . . . .	62
2.6.2. European put option . . . . .	63
2.6.3. American call option . . . . .	63
2.6.4. American put option . . . . .	64
2.6.5. Basic problems and assumptions . . . . .	65
2.7. Types of market participants . . . . .	67
2.7.1. Hedgers . . . . .	67
2.7.2. Speculators . . . . .	67
2.7.3. Arbitrageurs . . . . .	67
2.8. Arbitrage relationships between call and put options . . . . .	67
2.9. Exercises . . . . .	69
<b>Chapter 3. Conditional Expectation and Markov Chains . . . . .</b>	<b>71</b>
3.1. Introduction . . . . .	71
3.2. Conditional expectation: the discrete case . . . . .	72
3.3. Applications of conditional expectations . . . . .	75
3.3.1. Expectation of the sum of a random number of random variables . . . . .	76
3.3.2. Expected value of a random number of Bernoulli trials with probability of success being a random variable . . . . .	77
3.3.3. Number of Bernoulli trials until there are $k$ consecutive successes . . . . .	78
3.3.4. Conditional variance relationship . . . . .	79
3.3.5. Variance of the sum of a random number of random variables . . . . .	80
3.4. Properties of the conditional expectation . . . . .	81
3.5. Markov chains . . . . .	85
3.5.1. Probability distribution in the states of a Markov chain . . . . .	90
3.5.2. Statistical inference in Markov chains . . . . .	94
3.5.3. The strong Markov property . . . . .	97
3.5.4. Classification of states of a Markov chain . . . . .	100
3.5.5. Periodic Markov chains . . . . .	104

3.5.5.1. Cyclic subclasses . . . . .	106
3.5.5.2. Algorithm for the cyclic subclasses . . . . .	109
3.5.6. Classification of states . . . . .	112
3.5.7. Asymptotic behavior of irreducible homogenous Markov chains . . . . .	115
3.5.8. The mean time of first entrance in a state of Markov chain . . . . .	126
3.5.9. The variance of the time of first visit into a state of a Markov chain . . . . .	129
3.6. Exercises . . . . .	131
<b>Chapter 4. The No-Arbitrage Binomial Pricing Model . . . . .</b>	<b>137</b>
4.1. Introductory notes . . . . .	137
4.2. Binomial model . . . . .	138
4.3. Stochastic evolution of the asset prices . . . . .	141
4.4. Binomial approximation to the lognormal distribution . . . . .	143
4.5. One-period European call option . . . . .	145
4.6. Two-period European call option . . . . .	150
4.7. Multiperiod binomial model . . . . .	153
4.8. The evolution of the asset prices as a Markov chain . . . . .	154
4.9. Exercises . . . . .	158
<b>Chapter 5. Martingales . . . . .</b>	<b>163</b>
5.1. Introductory notes . . . . .	163
5.2. Martingales . . . . .	164
5.3. Optional sampling theorem . . . . .	169
5.4. Submartingales, supermartingales and martingales convergence theorem . . . . .	178
5.5. Martingale transforms . . . . .	182
5.6. Uniform integrability and Doob's decomposition . . . . .	184
5.6.1. Doob decomposition . . . . .	186
5.7. The snell envelope . . . . .	187
5.8. Exercises . . . . .	190
<b>Chapter 6. Equivalent Martingale Measures, No-Arbitrage and Complete Markets . . . . .</b>	<b>195</b>
6.1. Introductory notes . . . . .	195
6.2. Equivalent martingale measure and the Randon-Nikodým derivative process . . . . .	196
6.3. Finite general markets . . . . .	204
6.3.1. Uniqueness of arbitrage price . . . . .	210
6.3.2. Equivalent martingale measures . . . . .	213
6.4. Fundamental theorem of asset pricing . . . . .	215
6.5. Complete markets and martingale representation . . . . .	222

6.6. Finding the equivalent martingale measure . . . . .	228
6.6.1. Exploring the vital equations and conditions . . . . .	234
6.6.2. Equivalent martingale measures for general finite markets . . . . .	237
6.7. Exercises . . . . .	238
<b>Chapter 7. American Derivative Securities . . . . .</b>	<b>241</b>
7.1. Introductory notes . . . . .	241
7.2. A three-period American put option . . . . .	242
7.3. Hedging strategy for an American put option . . . . .	249
7.4. The algorithm of the American put option . . . . .	254
7.4.1. Algorithm of the American put option . . . . .	254
7.4.1.1. Pricing of the American put option . . . . .	254
7.4.1.2. Trading strategy for hedging . . . . .	254
7.5. Optimal time for the holder to exercise . . . . .	255
7.6. American derivatives in general markets . . . . .	262
7.7. Extending the concept of self-financing strategies . . . . .	266
7.8. Exercises . . . . .	269
<b>Chapter 8. Fixed-Income Markets and Interest Rates . . . . .</b>	<b>273</b>
8.1. Introductory notes . . . . .	273
8.2. The zero coupon bonds of all maturities . . . . .	274
8.3. Arbitrage-free family of bond prices . . . . .	278
8.4. Interest rate process and the term structure of bond prices . . . . .	282
8.5. The evolution of the interest rate process . . . . .	290
8.6. Binomial model with normally distributed spread of interest rates . . . . .	293
8.7. Binomial model with lognormally distributed spread of interest rates . . . . .	296
8.8. Option arbitrage pricing on zero coupon bonds . . . . .	298
8.8.1. Valuation of the European put call . . . . .	298
8.8.2. Hedging the European put option . . . . .	300
8.9. Fixed income derivatives . . . . .	302
8.9.1. Interest rate swaps . . . . .	304
8.9.2. Interest rate caps and floors . . . . .	307
8.10. $T$ -period equivalent forward measure . . . . .	308
8.11. Futures contracts . . . . .	317
8.12. Exercises . . . . .	319
<b>Chapter 9. Credit Risk . . . . .</b>	<b>323</b>
9.1. Introductory notes . . . . .	323
9.2. Credit ratings and corporate bonds . . . . .	324
9.3. Credit risk methodologies . . . . .	326
9.3.1. Structural methodologies . . . . .	326
9.3.2. Reduced-form methodologies . . . . .	327

9.4. Arbitrage pricing of defaultable bonds . . . . .	327
9.5. Migration process as a Markov chain . . . . .	330
9.5.1. Change of real-world probability measure to equivalent $T^*$ -forward measure . . . . .	331
9.6. Estimation of the real world transition probabilities . . . . .	334
9.7. Term structure of credit spread and model calibration . . . . .	337
9.8. Migration process under the real-world probability measure . . . . .	341
9.8.1. Stochastic monotonicities in default times . . . . .	344
9.8.2. Asymptotic behavior . . . . .	350
9.9. Exercises . . . . .	352
<b>Chapter 10. The Heath-Jarrow-Morton Model . . . . .</b>	<b>355</b>
10.1. Introductory notes . . . . .	355
10.2. Heath-Jarrow-Morton model . . . . .	356
10.2.1. Evolution of forward rate process . . . . .	356
10.2.2. Evolution of the savings account and short-term interest rate process . . . . .	358
10.2.3. Evolution of the zero-coupon non-defaultable bond process . . . . .	359
10.2.4. Conditions on the drift and volatility parameters for non-arbitrage . . . . .	360
10.3. Hedging strategies for zero coupon bonds . . . . .	362
10.4. Exercises . . . . .	364
<b>References . . . . .</b>	<b>365</b>
<b>Appendices . . . . .</b>	<b>374</b>
A. Appendix A . . . . .	375
A.1. Introductory thoughts . . . . .	375
A.2. Genesis . . . . .	376
A.3. The decisive steps . . . . .	378
A.4. A brief glance towards the flow of research paths . . . . .	387
B. Appendix B . . . . .	391
B.1. Introduction . . . . .	391
B.2. The main theorem . . . . .	392
<b>Index . . . . .</b>	<b>395</b>