Contents

List of Notations xi

Introduction xiii

Chapter 1. The Displacement Group as a Lie Group 1

1.1. General points 1
1.2. The groups $\mathrm{O}(\mathbb{E})$ and $\mathrm{SO}(\mathbb{E})$ as Lie groups 3
1.2.1. Preliminary remarks 3
1.2.2. Elementary calculus in $\mathrm{O}(\mathbb{E})$ and $\mathrm{SO}(\mathbb{E})$ seen as manifolds 7
1.2.3. Exponential mapping of $\mathrm{SO}(\mathbb{E})$ 12
1.3. The group $\mathbb{U}$ of normalized quaternions 16
1.3.1. Quaternionic representation of $\mathrm{SO}(\mathbb{E})$ 18
1.3.2. Complement 21
1.3.3. Angular velocity in quaternionic representation 24
1.4. Cayley transforms 25
1.4.1. Cayley transform defined on $\mathcal{L}_a(\mathbb{E})$ 25
1.4.2. Cayley transform defined on $\mathbb{E}$ 26
1.4.3. Relation between Cayley transform and quaternions 28
1.4.4. Angular velocity of a motion described with a Cayley representation 29
1.5. The displacement group as a Lie group 30
1.5.1. The displacement group as a matrix group 30
1.5.2. The displacement group as a group of affine maps 44
1.5.3. Classification of the Euclidean displacements 47
1.5.4. The Lie algebra of $\mathbb{D}$ as a Lie algebra of vector fields on $\mathcal{E}$ 49
1.5.5. The Klein form on $\mathfrak{D}$ 60
1.5.6. Operator ϵ 62

1.5.7. One-parameter subgroups of $\mathbb{D}$ and exponential mapping 66
1.6. Conclusion 71
1.7. Appendix 1: The algebra of quaternions 73
1.7.1. First definition of quaternions 73
1.7.2. Center of $\mathbb{H}$ 74
1.7.3. Conjugation in $\mathbb{H}$ 74
1.7.4. Euclidean structure of $\mathbb{H}$ 74
1.7.5. Second definition of quaternions 76
1.8. Appendix 2: Lie subalgebras and ideals of $\mathfrak{D}$ 77
1.8.1. Lie subgroups of $\mathbb{D}$ 79
1.8.2. Trivial Lie subgroups 81
1.8.3. One-parameter subgroups 81
1.8.4. Two-parameter subgroups 82
1.8.5. Three-parameter subgroups 82
1.8.6. Four-parameter subgroups 82
1.8.7. Five-parameter subgroups 83

Chapter 2. Dual Numbers and "Dual Vectors" in Kinematics 85

2.1. The Euclidean module $\mathfrak{D}$ over the dual number ring 86
2.1.1. The ring $\mathbf{\Delta}$ and the module structure on $\mathfrak{D}$ 86
2.1.2. Linear independence over $\mathbf{\Delta}$ 88
2.1.3. $\mathbf{\Delta}$-linear maps 89
2.1.4. Dual inner and mixed products 90
2.2. Dualization of a real vector space 97
2.2.1. General extension of a real vector space into a $\mathbf{\Delta}$-module 97
2.2.2. Dualization of the Euclidean vector space in dimension 3 99
2.2.3. The groups $\mathrm{O}(\mathfrak{D})$ and $\mathrm{SO}(\mathfrak{D})$ 101
2.2.4. Generalized Olinde Rodrigues formula 106
2.3. Dual quaternions 110
2.3.1. Geometrical definition 110
2.3.2. Norm and invertibility in $\mathcal{H}$ 111
2.3.3. Dual quaternions and representation of $\mathbb{D}$ in $\mathfrak{D}$ 112
2.4. Differential calculus in $\mathbf{\Delta}$-modules 113
2.4.1. $\mathbf{\Delta}$-differentiable maps 113
2.4.2. Extensions of ordinary differentiable maps into $\mathbf{\Delta}$-differentiable maps 114

Chapter 3. The "Transference Principle" 119

3.1. On the meaning of a general algebraic transference principle 119
3.2. Isomorphy between the adjoint group $\mathbb{D}_*$ and $\mathrm{SO}(\widehat{\mathbb{E}})$ 120
3.3. Regular maps 121
3.4. Extensions of the regular maps from U to $\mathrm{SO}(\mathbb{E})$ 123

Chapter 4. Kinematics of a Rigid Body and Rigid Body Systems 129

4.1. Introduction 129
4.2. Kinematics of a rigid body 131
4.3. The position space of a rigid body 131
4.4. Relations to the models of bodies 134
4.4.1. Example 1 134
4.4.2. Example 2 135
4.4.3. Fundamental theorem of kinematics of a rigid body 135
4.5. Changs of frame in kinematics 137
4.6. Graphs and systems subjected to constraints 139
4.6.1. A few elements of graph theory 139
4.6.2. The position space of a rigid body system 141
4.6.3. The various kinds of links between pairs of bodies 143
4.6.4. Kinematics of a linked pair of bodies 146
4.7. Kinematics of chains 148

Chapter 5. Kinematics of Open Chains, Singularities 153

5.1. The mathematical picture of an open chain 153
5.1.1. Articular coordinates 155
5.1.2. The shape function in articular coordinates 156
5.1.3. Further developments about the shape function 159
5.2. Singularities of a kinematic chain 164
5.2.1. General setting 164
5.2.2. A method to calculate the rank of a subset of $\mathfrak{D}$ 166
5.3. Examples: Singularities of open kinematic chains with parallel axes 167
5.3.1. Singularities of an open chain HHHH with parallel axes 168
5.3.2. Singularities of an open chain HPHP with parallel axes 175
5.3.3. Singularities of an open chain HPPH with parallel axes 183
5.3.4. Singularities of an open chain HHPP with parallel axes 186
5.3.5. Singularities of open chains HHH with parallel axes 188
5.3.6. Singularities of a chain HPH with parallel axes 190
5.4. Calculations of the successive derivatives of f 191
5.5. Transversality and singularities of a product of exponential mappings 194
5.5.1. Structure of manifold of E_r 195
5.5.2. Tangent space of E_r 197
5.5.3. Transversality 197
5.5.4. The tangent space of $\Sigma_r(f)$ at a weak singular point 201
5.5.5. The global features and the imperfections in the joints: **case** $n = \dim \mathfrak{D}$ 202

Chapter 6. Closed Kinematic Chains: Mechanisms Theory 207

6.1. Geometric framework and regular case 210
6.2. Exhaustive classification of the local singularities of mechanisms 215
6.2.1. Weak singular configuration 215
6.2.2. Strong singular configuration 216
6.3. Singular mechanisms with degree of mobility one 221
6.3.1. The principle of the method 223
6.3.2. The Benett mechanism 224
6.3.3. The Bricard mechanism 228
6.4. Concrete examples and calculations 230
6.4.1. The Bricard mechanism 231
6.4.2. The four-bar planar mechanism 232

Chapter 7. Dynamics 235

7.1. Changes of frame in dynamics, objective magnitudes 236
7.2. The inertial mass of a rigid body 240
7.2.1. Center of inertia 242
7.2.2. Bilinear form associated with the kinetic energy 243
7.2.3. Matrix representation of $\mathcal{H}_s$ 251
7.2.4. Kinetic and dynamic momentum 251
7.3. The fundamental law of dynamics 254
7.3.1. Mathematical pictures of forces 255
7.3.2. Statement of the fundamental law in Galilean frames 256
7.3.3. Energy integral 258
7.3.4. Law of dynamics with respect to a non-Galilean frame 260

Chapter 8. Dynamics of Rigid Body Systems 269

8.1. Systems subjected to constraints 269
8.1.1. The position space of a rigid body system 270
8.1.2. The various kinds of links between pairs of bodies 271
8.1.3. Forces exerted in links 277
8.2. The principles of dynamics for multibody systems 279
8.2.1. Newton-Euler form of the principles 279
8.2.2. Virtual power form of the principle 283
8.2.3. Virtual velocities in a system with a linkage 286
8.3. Tree-structured systems 287
8.3.1. Kinematics of a tree-structured system 288

8.3.2. Remarks 291
8.3.3. Virtual power of deformations in a system with a linkage 292
8.3.4. Application of the principle of virtual works to a tree structured system 294
8.4. Complement: Lagrange's form of the virtual power of the inertial forces 298
8.5. Appendix: The subspaces $\mathfrak{n}(s)$ and $\mathfrak{m}(s)$ associated with the Lie subalgebras of $\mathfrak{D}$ 301
8.5.1. Dimension 1 Lie subalgebras and Lie subgroups 302
8.5.2. Dimension 2 Lie subalgebras and Lie subgroups 302
8.5.3. Dimension 3 Lie subalgebras and Lie subgroups 303
8.5.4. Dimension 4 Lie subalgebras and Lie subgroups 303

Bibliography 305

Index 313