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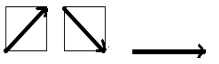
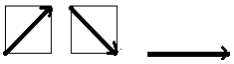
10.1. Mountain ranges based on three lines  197

10.2. Words based on three lines  with as many rising lines as falling lines 198

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