Contents

Preface ix

List of Notations xiii

Chapter 1. Categories and Functors 1

1.1. Categories 1
1.1.1. General results about categories 1
1.1.2. The category of sets 5
1.2. Functors 13
1.2.1. Covariant functors and contravariant functors 13
1.2.2. Functorial morphisms 14
1.2.3. Bifunctor Hom 15
1.2.4. Universal arrows and universal elements 16
1.2.5. Representable functors 18
1.2.6. Products and coproducts 19
1.2.7. Fibered products and fibered coproducts 20
1.2.8. Inductive limits and projective limits 21
1.2.9. Exact functors and adjoint functors 25
1.2.10. Projective objects and injective objects 26
1.2.11. Generators and cogenerators 28
1.3. Structures 28
1.3.1. Concrete categories 28
1.3.2. Initial structures and terminal structures 29
1.3.3. Concrete functors 30
1.3.4. Free objects, free functor 30

Chapter 2. Elementary Algebraic Structures 33

2.1. Monoids and ordered sets 34
2.1.1. Monoids and divisibility 34
2.1.2. Ordered sets 36
2.1.3. Lattice 37
2.2. Groups 40
2.2.1. Groups and subgroups 40
2.2.2. Normal subgroups 42
2.2.3. Fundamental isomorphisms 44
2.2.4. Cyclic groups and simple groups 46
2.2.5. Lattice of normal subgroups 47
2.2.6. Derived subgroup 48
2.2.7. Solvable groups and nilpotent groups 49
2.2.8. Action of a group on a set 51
2.3. Rings and algebras 52
2.3.1. Rings and modules 52
2.3.2. Ideals 54
2.3.3. Maximal ideals and prime ideals. Spectrum 58
2.3.4. Noetherian rings and Artinian rings 60
2.3.5. Division rings. Simple modules and simple Artinian rings. Radical 61
2.3.6. Nilradical. Radical ideal 66
2.3.7. Local rings 67
2.3.8. Principal ideal domains and related notions 67
2.3.9. Polynomial rings and formal power series rings 74
2.3.10. General notions relating to algebras 77
2.3.11. Matrix algebras and determinants 79
2.3.12. Graded algebras and graded modules 84

Chapter 3. Modules and Algebras 87

3.1. Additional concepts from linear algebra 89
3.1.1. Bimodules 89
3.1.2. Duality 89
3.1.3. Free modules 90
3.1.4. Exact sequences 94
3.1.5. Tensor products 96
3.1.6. Generators and relations 102
3.1.7. Modules over Noetherian rings 106

3.1.8. $\mathfrak{m}$-adic completion 107
3.1.9. Ring of fractions 109
3.1.10. Division rings of fractions 112
3.1.11. Polynomial rings and skew Laurent polynomials 118
3.2. Notions of commutative algebra 122
3.2.1. Localization at a prime ideal 122
3.2.2. Notions of algebraic geometry and number theory 125
3.2.3. Supp and Ass 128
3.2.4. Primary decomposition 131
3.2.5. Elements integral over a ring. *Nullstellensatz* 136
3.2.6. Krull dimension 143
3.2.7. Algebraic sets 146
3.3. Homological notions 149
3.3.1. Projective modules and injective modules 149
3.3.2. Malgrange isomorphism 155
3.3.3. Injective cogenerators 159
3.3.4. Dedekind domains 165
3.3.5. Global dimension 169
3.3.6. Bézout equations 173
3.3.7. Preabelian caregories and abelian categories 175
3.3.8. Complexes. Notions of algebraic topology 183
3.3.9. Derived functors 198
3.4. Modules over principal ideal domains and related notions 202
3.4.1. Modules over Bézout domains 202
3.4.2. Modules over principal ideal domains 205
3.4.3. Pseudo-linear transformations 213
3.4.4. Systems of linear differential equations with constant coefficients 220
3.4.5. Modules over Dedekind domains 222

Bibliography 225

Cited Authors 229

Index 235