
Contents

Preface	ix
Chapter 1. Theoretical Aspects of Elliptic Equations	1
1.1. Some vocabulary	1
1.1.1. Boundary conditions for the Laplace equation	3
1.1.2. Other boundary conditions	4
1.2. Classical solutions to Laplace's equation	5
1.2.1. Green's formulae	5
1.2.2. The maximum principle	8
1.3. Weak solutions	9
1.4. Reminders on Banach and Hilbert spaces	11
1.4.1. Banach spaces	12
1.4.2. Hilbert spaces	13
1.5. Sobolev spaces	16
1.5.1. Differentiation in the distribution sense	17
1.5.2. "Sufficiently regular" boundaries	23
1.5.3. Differentiation in the classical or distribution sense?	25
1.5.4. A compactness result	26
1.6. Existence of the trace and integration by parts formulae	27
1.7. Exercises – statements	35
Chapter 2. Variational Formulations and Their Solutions	43
2.1. Variational formulations of the Dirichlet, Neumann and Fourier problems	43
2.1.1. Homogeneous Dirichlet problem	43
2.1.2. Non-homogeneous Dirichlet problem	46
2.1.3. Neumann and Fourier problems	47
2.2. Other variational formulations examples	50

2.2.1. Variable coefficient problems	50
2.2.2. Second-order elliptic problems	51
2.2.3. Higher order problems	51
2.2.4. Less-regular data	52
2.3. Existence and uniqueness of weak solutions	53
2.3.1. Neumann problem	54
2.3.2. Dirichlet problem – Poincaré’s inequality	56
2.3.3. The non-homogeneous Dirichlet problem	59
2.4. Existence and uniqueness – general framework	60
2.4.1. Lax–Milgram theorem	61
2.4.2. An example application of the Lax–Milgram theorem with the Laplacian	64
2.4.3. The Helmholtz problem	67
2.4.4. Well-posed problems	71
2.4.5. The Banach–Necas–Babuška theorem	72
2.4.6. Fredholm alternative	76
2.5. Some properties of weak solutions	78
2.5.1. Maximum principle	79
2.5.2. Regularity of the solutions	81
2.6. Exercises – statements	85
Chapter 3. Introduction to the Finite Element Method	95
3.1. Galerkin’s approximation	95
3.1.1. Case of a coercive form	97
3.1.2. Case of a form satisfying the stability and solvability conditions .	101
3.1.3. Approximation example: Hilbert bases	104
3.1.4. Different solution and test-function spaces	105
3.2. Principles of the affine finite element method in two dimensions	107
3.2.1. Mesh	107
3.2.2. Global basis functions properties	110
3.2.3. Norm of the elements of the approximation space	112
3.2.4. Sparse matrix property	113
3.2.5. Lagrange interpolation	113
3.2.6. Local nature of the finite element method	113
3.3. General procedure of constructing finite elements	117
3.3.1. Lagrangian finite elements of order k	117
3.3.2. Examples of finite elements	118
3.3.3. Constructing the approximation space	127
3.3.4. Two-dimensional examples of assembly	129
3.4. Extension of finite elements	133
3.4.1. Hermitian finite elements	134
3.4.2. “Moment”-type finite elements	135
3.4.3. Constructing the approximation space	136

3.4.4. Vectorial finite elements	137
3.5. Exercises – statements	137
Chapter 4. Numerical Analysis of the Finite Element Method	143
4.1. Convergence of finite element methods	143
4.1.1. Local interpolation error	145
4.1.2. Estimating errors in the finite element method	152
4.1.3. “Piecewise regular” solutions	156
4.2. Error estimators and mesh refinement	160
4.3. Non-polyhedral domains and approximations of the data	163
4.3.1. Non-polyhedral open subsets	163
4.3.2. Isoparametric finite elements	167
4.3.3. Approximation of the data	168
4.4. Exercises – statements	172
Chapter 5. Concrete Aspects of the Finite Element Method	185
5.1. Implementation	185
5.1.1. Finite element meshes	186
5.1.2. Basic calculations	189
5.1.3. Assembly of the global matrices and the right-hand side	191
5.1.4. Eliminating the essential boundary conditions	195
5.2. Algorithmic considerations	199
5.2.1. A particular case	200
5.2.2. The general 2D case	200
5.2.3. The general 3D case	205
5.3. Some numerical illustrations	207
5.3.1. The Laplace equation	208
5.3.2. The diffusion equation	213
5.3.3. Bidimensional elasticity	215
5.3.4. Some elementary tools for 2D meshes	222
Appendices	231
Appendix 1. Solving Linear Systems	233
Appendix 2. Solutions	271
Appendix 3. Formulas	369
References	371
Index	373