
Contents

Foreword	xiii
Preface	xv
Part 1. Simulation and Identification of Fractional Differential Equations (FDEs) and Systems (FDSs)	1
Chapter 1. The Fractional Integrator	3
1.1. Introduction	3
1.2. Simulation and modeling of integer order ordinary differential equations	3
1.2.1. Simulation with analog computers	3
1.2.2. Simulation with digital computers.	5
1.2.3. Initial conditions	6
1.2.4. State space representation and simulation diagram	7
1.2.5. Concluding remarks	9
1.3. Origin of fractional integration: repeated integration.	10
1.4. Riemann–Liouville integration	12
1.4.1. Definition	12
1.4.2. Laplace transform of the Riemann–Liouville integral	13
1.4.3. Fractional integration operator.	14
1.4.4. Fractional differentiation	15
1.5. Simulation of FDEs with a fractional integrator	17
1.5.1. Simulation of a one-derivative FDE.	17
1.5.2. FDE	18
1.5.3. Simulation of the general linear FDE	18
A.1. Appendix.	20
A.1.1. Lord Kelvin’s principle	20

A.1.2. A brief history of analog computing	21
A.1.3. Interpretation of the RK2 algorithm	22
A.1.4. The gamma function	23
Chapter 2. Frequency Approach to the Synthesis of the Fractional Integrator	25
2.1. Introduction	25
2.2. Frequency synthesis of the fractional derivator	26
2.3. Frequency synthesis of the fractional integrator	28
2.3.1. Objective.	28
2.3.2. Direct method	29
2.3.3. Indirect method	30
2.3.4. Frequency synthesis of $1/s^n$	32
2.4. State space representation of $I_d^n(s)$	33
2.5. Modal representation of $I_d^n(s)$	36
2.6. Numerical algorithm.	40
2.7. Frequency validation	41
2.8. Time validation	44
2.9. Internal state variables.	47
A.2. Appendix: design of fractional integrator parameters	49
A.2.1. Definition of G_n	49
A.2.2. Definition of α and η	51
Chapter 3. Comparison of Two Simulation Techniques	55
3.1. Introduction	55
3.2. Simulation with the Grünwald–Letnikov approach.	56
3.2.1. Euler’s technique	56
3.2.2. The Grünwald–Letnikov fractional derivative	58
3.2.3. Numerical simulation with the Grünwald–Letnikov integrator	60
3.2.4. Some specificities of the Grünwald–Letnikov integrator	61
3.2.5. Short memory principle	63
3.3. Simulation with infinite state approach	66
3.4. Caputo’s initialization.	68
3.5. Numerical simulations.	69
3.5.1. Introduction	69
3.5.2. Comparison of discrete impulse responses (DIRs).	70
3.5.3. Simulation accuracy	72
3.5.4. Static error caused by the short memory principle.	74
3.5.5. Caputo’s initialization	75
3.5.6. Conclusion.	78

A.3. Appendix: Mittag-Leffler function	78
A.3.1. Definition	78
A.3.2. Laplace transform	79
A.3.3. Unit step response of $1/(s^n + a)$	79
A.3.4. Caputo's initialization	80
Chapter 4. Fractional Modeling of the Diffusive Interface	81
4.1. Introduction	81
4.2. Heat transfer and diffusive model of the plane wall	82
4.2.1. Heat transfer.	82
4.2.2. Physical model of the diffusive interface	83
4.2.3. Frequency analysis of the diffusive phenomenon	85
4.2.4. Time analysis of the diffusive phenomenon	86
4.2.5. Conclusion.	87
4.3. Fractional commensurate order models	88
4.3.1. Physical origin	88
4.3.2. Analysis of physical commensurate order models	89
4.4. Optimization of the fractional commensurate order model	91
4.4.1. The proposed frequency approach.	91
4.4.2. Conclusion.	96
4.5. Fractional non-commensurate order models.	97
4.5.1. Justification	97
4.5.2. Parameter estimation of $H_{n_1, n_2}(s)$	97
4.5.3. Numerical examples	98
4.5.4. Conclusion.	101
4.6. Conclusion	102
A.4. Appendix: estimation of frequency responses – the least-squares approach	102
A.4.1. Identification of the commensurate order model $H_{N-1, N}(j\omega)$	103
A.4.2. Parameter estimation of the non-commensurate model $H_{n_1, n_2}(j\omega)$	104
Chapter 5. Modeling of Physical Systems with Fractional Models: an Illustrative Example	107
5.1. Introduction	107
5.2. Modeling with mathematical models: some basic principles	108
5.3. Modeling of the induction motor	109
5.3.1. Construction of the induction motor.	109
5.3.2. Principle of operation.	109
5.3.3. Induction motor knowledge model	110
5.3.4. Park's model.	112
5.3.5. Fractional Park's model	114

5.4. Identification of fractional Park's model	117
5.4.1. Simplified model	117
5.4.2. Identification algorithm	118
5.4.3. Nonlinear optimization	119
5.4.4. Simulation of \hat{y}_k and $\underline{\sigma}_k$	122
5.4.5. Comments	122
5.4.6. Application to the identification of fractional Park's model	123
Part 2. The Infinite State Approach	127
Chapter 6. The Distributed Model of the Fractional Integrator	129
6.1. Introduction	129
6.2. Origin of the frequency distributed model	130
6.3. Frequency distributed model	133
6.4. Finite dimension approximation of the fractional integrator	134
6.5. Frequency synthesis and distributed model	136
6.6. Numerical validation of the distributed model	138
6.6.1. Reconstruction of the weighting function	138
6.6.2. Reconstruction of the impulse response	140
6.7. Riemann–Liouville integration and convolution	142
6.7.1. Conclusion	147
6.8. Physical interpretation of the frequency distributed model	147
6.8.1. The infinite RC transmission line	147
6.8.2. RC line and spatial Fourier transform	149
6.8.3. Impulse response of the RC line	151
6.8.4. General solution	153
6.8.5. Initialization in the time and spatial domains	155
A.6. Appendix: inverse Laplace transform of the fractional integrator	156
Chapter 7. Modeling of FDEs and FDSs	159
7.1. Introduction	159
7.2. Closed-loop modeling of an elementary FDS	160
7.3. Closed-loop modeling of an FDS	162
7.3.1. Modeling of an N-derivative FDS	162
7.3.2. Distributed state	165
7.4. Transients of the one-derivative FDS	168
7.4.1. Numerical simulation	168
7.4.2. Initialization at $t = t_1$	169
7.4.3. Initialization at different instants	171
7.5. Transients of a two-derivative FDS	173

7.6. External or open-loop modeling of commensurate fractional order FDSs.	175
7.6.1. Introduction	175
7.6.2. External model of an elementary FDE	176
7.6.3. External representation of a two-derivative FDE.	179
7.6.4. External representation of an N-derivative FDE	180
7.7. External and internal representations of an FDS	182
7.8. Computation of the Mittag-Leffler function.	183
7.8.1. Introduction	183
7.8.2. Divergence of direct computation	184
7.8.3. Step response approach.	185
7.8.4. Improved step response approach	186
A.7. Appendix: inverse Laplace transform of $1/(s^n + a)$	189
Chapter 8. Fractional Differentiation	193
8.1. Introduction	193
8.2. Implicit fractional differentiation.	194
8.3. Explicit Riemann–Liouville and Caputo fractional derivatives	195
8.3.1. Definitions.	195
8.3.2. Theoretical prerequisites	197
8.3.3. Comments	198
8.4. Initial conditions of fractional derivatives	199
8.4.1. Introduction	199
8.4.2. Implicit derivative	200
8.4.3. Caputo derivative	201
8.4.4. Riemann–Liouville derivative	203
8.4.5. Relations between initial conditions.	204
8.5. Initial conditions in the general case	205
8.5.1. Introduction	205
8.5.2. Implicit derivatives	205
8.5.3. Caputo derivatives	206
8.5.4. Riemann–Liouville derivatives	207
8.5.5. Relations between initial conditions.	207
8.6. Unicity of FDS transients	208
8.6.1. Transients of the elementary FDE.	208
8.6.2. Unicity of transients	209
8.6.3. Conclusion.	210
8.7. Numerical simulation of Caputo and Riemann–Liouville transients	212
8.7.1. Introduction	212
8.7.2. Simulation of Caputo derivative initialization	212
8.7.3. Simulation of Riemann–Liouville initialization	215

Chapter 9. Analytical Expressions of FDS Transients	219
9.1. Introduction	219
9.2. Mittag-Leffler approach	221
9.2.1. Free response of the elementary FDS	221
9.2.2. Free response of the N-derivative FDS	223
9.2.3. Complete solution of the N-derivative FDS	225
9.3. Distributed exponential approach	227
9.3.1. Introduction	227
9.3.2. Solution of $D^n(x(t)) = ax(t)$ using frequency discretization	227
9.3.3. Solution of $D^n(x(t)) = ax(t)$ using a continuous approach	230
9.3.4. Solution of $D^n(x(t)) = ax(t)$ using Picard's method	232
9.3.5. Solution of $D^\alpha(\underline{X}(t)) = A\underline{X}(t)$	235
9.3.6. Solution of $D^\alpha(\underline{X}(t)) = A\underline{X}(t) + \underline{B}u(t)$	237
9.4. Numerical computation of analytical transients	237
9.4.1. Introduction	237
9.4.2. Computation of the forced response	238
9.4.3. Step response of a three-derivative FDS	240
Chapter 10. Infinite State and Fractional Differentiation of Functions	243
10.1. Introduction	243
10.2. Calculation of the Caputo derivative	244
10.2.1. Fractional derivative of the Heaviside function	245
10.2.2. Fractional derivative of the power function	246
10.2.3. Fractional derivative of the exponential function	248
10.2.4. Fractional derivative of the sine function	249
10.3. Initial conditions of the Caputo derivative	250
10.4. Transients of fractional derivatives	253
10.4.1. Introduction	253
10.4.2. Heaviside function	254
10.4.3. Power function	255
10.4.4. Exponential function	256
10.4.5. Sine function	256
10.5. Calculation of fractional derivatives with the implicit derivative	257
10.5.1. Introduction	257
10.5.2. Fractional derivative of the Heaviside function	258
10.5.3. Fractional derivative of the power function	259
10.5.4. Fractional derivative of the exponential function	260
10.5.5. Fractional derivative of the sine function	261
10.5.6. Conclusion	262

10.6. Numerical validation of Caputo derivative transients	262
10.6.1. Introduction	262
10.6.2. Simulation results	264
A.10. Appendix: convolution lemma	266
References	269
Index	285