Chapter 1

Discrete Geometry and Projections

1.1. Introduction

This book is devoted to a discrete Radon transform named the Mojette transform. The Radon transform specificity is to mix Cartesian and radial views of the plane. However, it is straightforward to obtain a discrete lattice from a Cartesian grid while it is impossible from a standard equiangular radial grid. The only mathematical tool is to use discrete geometry that replaces the equiangular radial line by discrete lines aligned with a pixel grid. This chapter presents and investigates these precious tools.

After having examined the structure of the discrete space in section 1.2, we will focus on topological (section 1.3) and arithmetic (section 1.4) principles that guide the definition of discrete geometry elements (section 1.5).

1.2. Discrete pavings and discrete grids

First of all, let us define the underlying domain: without loss of generality, we can define the *discrete space* as a tiling P of the plane (or the space in higher dimensions) with non-overlapping enumerable open cells. Hence, a *discrete representation* of a continuous function $f : \mathbb{R}^2 \rightarrow I$ is a set of valued

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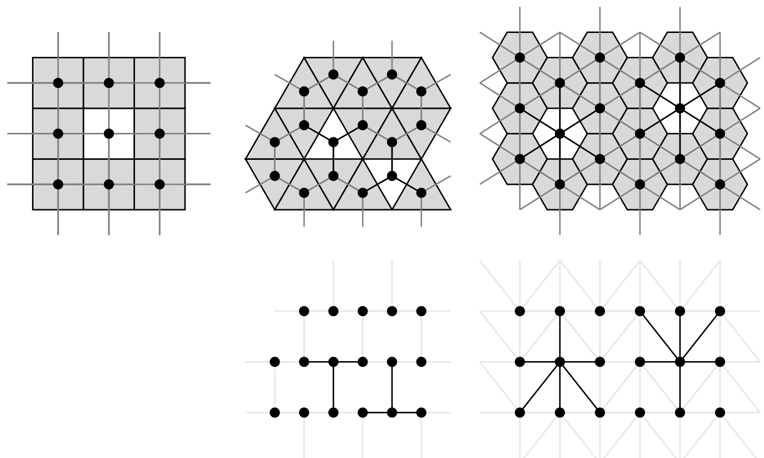

Figure 1.1. *Regular grids and regular tilings in two dimensions*

cells of P. The function that associates points in $\mathbb{R}^2$ with cells of P and defines a transfer function for the values I is called a *digitization process*.

Instead of considering the tiling of cells, we may also consider the dual representation, referred to as the *discrete grid*. A cell is represented by a single point (usually its center of gravity) and two points are connected if the two associated closed cells share an edge or a vertex. Such points are called *discrete points*.

In computer vision, regular tiling is preferred for many reasons: easy storage of cells; easy access to neighboring cells; coordinates can be mapped to $\mathbb{Z}^n$, closer to image capture devices such as charge-coupled devices (CDD). In two dimensions (2D), three regular tilings exist (Figure 1.1). We can note that even if cell shapes and cell adjacencies seem to be complex for the triangular and hexagonal tilings, vertices of their associated discrete grids can be easily mapped to the $\mathbb{Z}^2$ coordinate system.

In three dimensions, a classical regular grid is the cubic grid, composed of *voxels* whose centers can be mapped into $\mathbb{Z}^3$. In the rest of this chapter, we only focus on a 2D grid. Note that if some elements presented below can be easily extended to 3D, the generalization to a higher dimension of geometrical and topological principles is usually complex.

1.3. Topological principles and discrete objects

From cell adjacencies we can define elementary objects from which topological properties can be derived. First of all, let us consider two points A and B in the square discrete grid. Since A and B can be mapped to $\mathbb{Z}^2$ (coordinates associated with A are denoted with subscripts (x_A, y_A)), A and B are 4-*adjacent* if

$$|x_A - x_B| + |y_A - y_B| = 1\,.$$

In other words, closed cells associated with A and B share an edge and A therefore has exactly 4 neighbors. We can also define the 8-*adjacency* as follows: A and B are 8-*adjacent* if

$$\max\left(|x_A - x_B|, |y_A - y_B|\right) = 1\,.$$

Based on this definition, closed cells of A and B share either an edge or a vertex.

In the hexagonal grid, only one adjacency relationship exists since neighboring closed cells sharing a vertex also share an edge (and conversely). If we consider the coordinate mapping into $\mathbb{Z}^2$ of the hexagonal discrete points (Figure 1.1), neighbors of a cell (i, j) are

$$\left\{\, (i-1, j)\,,\, (i, j-1)\,,\, (i, j+1)\,,\, (i+1, j-1)\,,\, (i+1, j)\,,\, (i+1, j+1)\,\right\}$$

if j is even, and

$$\left\{\, (i-1, j-1)\,,\, (i-1, j)\,,\, (i-1, j+1)\,,\, (i, j-1)\,,\, (i, j+1)\,,\, (i+1, j)\,\right\}$$

otherwise. Similar embedding of the triangular grid discrete points into $\mathbb{Z}^2$ is possible.

The adjacency relationships presented above are such that the closed cells of neighboring grid points share either an edge or a vertex. Many other relationships can be defined if we remove the constraint on the closed cells. However, since many topological results depend on the adjacency relationship, many of them may not exist for such general adjacencies.

Considering the relationships presented above, several elementary objects can be defined. For the sake of simplicity, we consider the k-adjacency as the abstraction of the possible adjacencies.

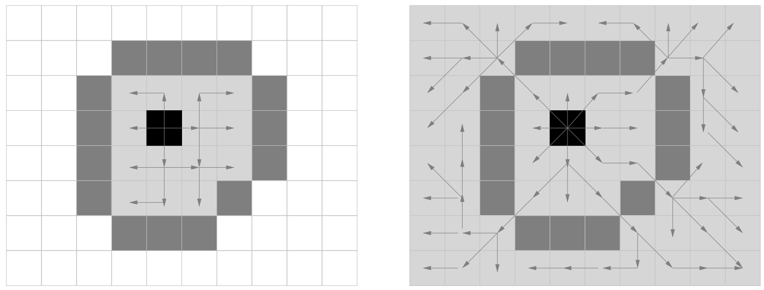

Figure 1.2. *Adjacency relationship pairs illustrated with an object-filling process*

DEFINITION 1.1.– *Let X be a set of discrete points and a k-adjacency relationship. Then*

– *a k-*path *in X between two points A and B $(A, B \in X)$ is a sequence $\pi = (A_0, \ldots, A_n)$ such that $A_0 = A$, $A_n = B$, $A_i \in X$ and such that A_i is k-adjacent to A_{i-1} (for $i = 1, \ldots, n$);*

– *X is a k-*object *(or a k-*connected component*) if there exists a k-path in X between each couple A and B of points in X;*

– *if π is a k-path, π is a k-*closed curve *if each point in π has exactly two neighbors in π;*

– *a k-*curve *is a k-path π such that each point in π has exactly two neighbors in π, except for A_0 and A_n (also called* curve extremities*).*

On incorporation of these trivial definitions, many topological problems arise when we want to consider the border (or boundary) of a discrete object. Basically, in 2D, we would like the boundary of a discrete object to be a closed curve as defined above, such that it decomposes the plane into an interior (bounded) and an exterior (unbounded) domain. This property is called the *Jordan* property due to the Jordan Theorem in differential mathematics.

Let us illustrate this property with a simple curve-filling problem (Figure 1.2). The problem is to fill the interior of the gray discrete curve starting from the black cell. We first notice that the discrete curve is a 8-closed curve. If we choose the 4-adjacency in the filling process, we correctly fill the curve. On the other hand, if we consider the 8-adjacency, the filling process will cover the complete domain. In some sense, the discrete curve of this second option does not have the Jordan property.

In 2D and for the square grid, we need to handle two adjacencies relationships: one for the object and the other from the background. The Jordan property can be proven if the considered pair is either $(4, 8)$ or $(8, 4)$.

Finally, note that such problems are even more complex in higher dimensions.

1.4. Arithmetic principles

Whatever the regular grid we consider, vertices can be mapped into $\mathbb{Z}^2$ in 2D. As we will see in the following sections, many geometrical properties will be a consequence of arithmetic principles on integer numbers. In this section, we give an overview of some elements.

1.4.1. *Preliminaries*

Let us consider the set of integer numbers $\mathbb{Z}$. Recall that $\mathbb{Z}$ equipped with the operators $'+'$ and $'\cdot'$ is a commutative ring.

1) $(\mathbb{Z},' +')$ is an Abelian group ($+$ is associative, commutative, possesses an identity element and each integer has an inverse for this operator).

2) $(\mathbb{Z},' \cdot')$ is a monoid (associative, with an identity element).

3) Note that $'\cdot'$ distributes over $'+'$.

Many arithmetic results will be induced from the fact that $'\cdot'$ has no inverse on $\mathbb{Z}$ (in other words, $\mathbb{Z}$ is not a field). Despite this, we can define the division as follows: given $a, b \in \mathbb{Z}$, a *divides* b if there exists $c \in \mathbb{Z}$ such that $a \cdot c = b$ (a is also called a divisor of b). With the same condition, we can say that b is an *integer multiple* of a.

We next define the congruence relation between integers: given three integers $a, b, c \in \mathbb{Z}$, a is *congruent* to b modulo c (denoted $a \equiv b \pmod{c}$) if $a - b$ is an integer multiple of c (i.e. there exist $k \in \mathbb{Z}$ such that $a - b = kc$). Given an integer number c, the congruence relation modulo c over $\mathbb{Z}$ defines a set of c equivalence classes, denoted $\mathbb{Z}/c\mathbb{Z}$. Each equivalence class is denoted $[\![a]\!]_c$ (with $a \in \{0, \ldots, c-1\}$). In other words, $m, n \in [\![a]\!]_c$ if $m \equiv a \pmod{c}$ and $n \equiv a \pmod{c}$. Given an integer c, $\mathbb{Z}/c\mathbb{Z}$ is also a commutative ring.

Prime numbers will be crucial for many arithmetic properties in discrete geometry. An integer number $a > 1$ is prime if its divisors are either 1 or a itself. Note that if c is prime, each integer $a \in \mathbb{Z}/c\mathbb{Z}$ different from zero admits an inverse for the $'\cdot'$ operator. In other words, $\mathbb{Z}/c\mathbb{Z}$ becomes a field. In this book, we do not go further into detail on prime number theory. Interested readers can refer to [HAR 75].

Given two numbers a and b, the *greatest common divisor* of a and b denoted $\gcd(a, b)$ is the greatest positive number c such that c divides both a and b. Since any number divides 0, $\gcd(0, a) = a$. The *least common multiple*, $\mathrm{lcm}(a, b)$, is the least positive number c such that c is a multiple of both a and b.

Considering the greatest common divisor, we recall the Bézout identity that can be stated as follows: given $a, b \in \mathbb{Z}$, there exist $m, n \in \mathbb{Z}$ such that

$$am + bn = \gcd(a, b)\,.$$

From $\mathbb{Z}$, we can derive the set of *rational numbers* $\mathbb{Q} = m/n, m, n \in \mathbb{Z}, n \neq 0\}$ which is a field for the $'+'$ and $'\cdot'$ operators. A fraction m/n is called irreducible if $\gcd(m, n) = 1$. Note that if $\gcd(m, n) = 1$, these two numbers are also said to be *relatively prime* or *coprime*.

1.4.2. *Lattices*

In 2D, let us consider two vectors $\vec{u}, \vec{v} \in \mathbb{Z}^2$ and let O be a discrete point. The set $\mathcal{L}$ of discrete points defined by:

$$\vec{OP} = \lambda\vec{u} + \kappa\vec{v} \qquad (1.1)$$

with $\lambda, \kappa \in \mathbb{Z}$, is called a *lattice*. Considering that if $\vec{u}$ and $\vec{v}$ are not colinear, such a lattice is a vector subspace over $\mathbb{Z}^2$ (2D) whose *base vectors* are $\vec{u}$ and $\vec{v}$ (Figure 1.3); $\left\{\vec{u},\vec{v}\right\}$ is called the *lattice base* of $\mathcal{L}$.

The points P generated by equation (1.1) are called the *lattice points* generated by $\vec{u}$ and $\vec{v}$. Several pairs of vectors $\vec{u}$ and $\vec{v}$ may generate the same set of lattice points. Such lattices are said to be *equivalent*. For example, the lattices $\{O, \vec{u}, \vec{v}\}$, $\{O, -\vec{u}, \vec{v}\}$ and $\{O, \vec{u}, \vec{u} + \vec{v}\}$ are equivalent.

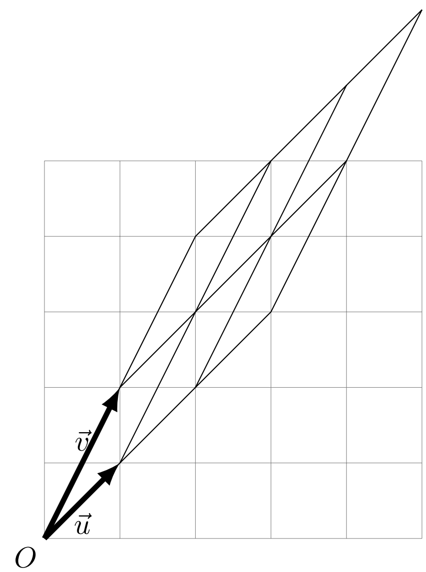

Figure 1.3. *Subset of a lattice* $\{O, \vec{u} = (1,1)^T, \vec{v} = (1,2)^T\}$

The lattice defined by the unitary vectors $\vec{u} = (1,0)^T$ and $\vec{v} = (0,1)^T$ is called the *fundamental lattice*. Let us denote the set of lattice points generated by the fundamental lattice by Λ, i.e. $\Lambda = \mathbb{Z}^2$.

If we consider $\vec{u} = (a, c)$ and $\vec{v} = (b, d)$, an important result can be stated as follows.

THEOREM 1.1.– *[HAR 75] The lattice points generated by* $\vec{u}$ *and* $\vec{v}$ *is equivalent to the fundamental lattice* Λ *if and only if det*$(\vec{u}, \vec{v}) = ad - bc = \pm 1$. Such numbers m and n can be determined using the extended Euclidean algorithm.

In other words, if the area of the parallelogram defined by $\{O, \vec{u}, \vec{v}\}$ is equal to 1, then the lattice $\{O, \vec{u}, \vec{v}\}$ spans $\mathbb{Z}^2$. In this case, $\vec{u}$ and $\vec{v}$ are called *unimodular*. For example, the lattice illustrated in Figure 1.3 is equivalent to Λ. Given a vector $\vec{u} = (a, b)$ with $a, b \in \mathbb{Z}$ such that $\gcd(a, b) = 1$, the Bézout identity of a and b presented in section 1.4.1 defines the vector $\vec{v} = (m, n)$ such that $\vec{u}$ and $\vec{v}$ are unimodular.

1.4.3. *Farey series, rational angle and Stern–Brocot tree*

First of all, let us consider a classical object in number theory: the *Farey series* [HAR 75].

DEFINITION 1.2.– *The* Farey series $\mathcal{F}_m$ *of order* m *is the ascending series of irreducible fractions between* 0 *and* 1 *whose denominators do not exceed* m.

For example, the Farey series of order 5 is:

$$\mathcal{F}_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}$$

Such a series has the following properties:

– if $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms in $\mathcal{F}_m$ (with $\frac{h}{k} < \frac{h'}{k'}$) , then $kh' - hk' = 1$;

– if $\frac{h}{k}$, $\frac{h''}{k''}$ and $\frac{h'}{k'}$ are three successive terms in $\mathcal{F}_m$ (with $\frac{h}{k} < \frac{h''}{k''} < \frac{h'}{k'}$), then $\frac{h''}{k''} = \frac{h+h'}{k+k'}$ (the fraction $\frac{h''}{k''}$ is called the *mediant* of $\frac{h}{k}$ and $\frac{h'}{k'}$).

Finally, the Farey series $\mathcal{F}_{m+1}$ can be computed from $\mathcal{F}_m$ adding the mediant with denominator less than or equal to $m+1$ of each two successive fractions in $\mathcal{F}_m$. For example, $\mathcal{F}_6$ is obtained from $\mathcal{F}_5$ adding the following bold terms:

$$\mathcal{F}_6 = \left\{ \frac{0}{1}, \frac{\mathbf{1}}{\mathbf{6}}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{\mathbf{5}}{\mathbf{6}}, \frac{1}{1} \right\}$$

The Stern–Brocot tree is a binary tree starting with $\frac{0}{1}$ and $\frac{1}{1}$ and iteratively inserting the mediant between two successive terms $\frac{h}{k}$ and $\frac{h'}{k'}$ (Figure 1.4). Another definition can be stated as follows: given a node $\frac{h}{k}$ and its father $\frac{h'}{k'}$ in the tree (without loss of generality $\frac{h'}{k'} < \frac{h}{k}$), then $\frac{h}{k}$ is the mediant between its father ($\frac{h'}{k'}$) and its first ancestor greater than $\frac{h}{k}$. In other words, the Stern–Brocot tree can be interpreted as a structure over Farey series.

1.4.4. *Geometrical interpretations of arithmetic results*

Many links exist between irreducible fractions and lattices. Most importantly, two consecutive fractions $\frac{h}{k}$ and $\frac{h'}{k'}$ are such that $kh' - hk' = 1$. Then

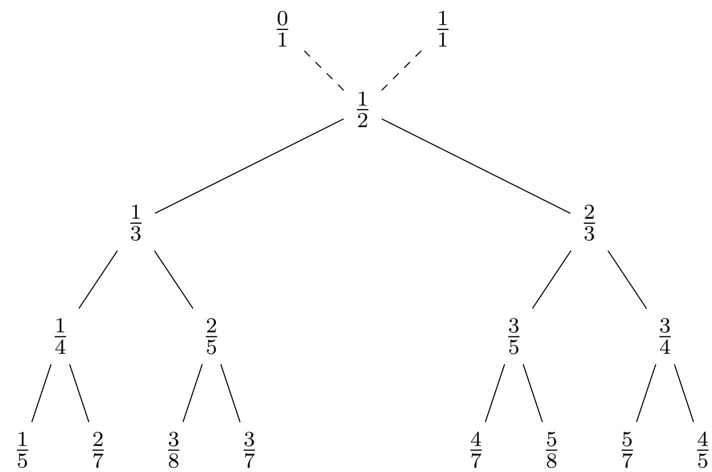

Figure 1.4. *Stern–Brocot tree of $\mathcal{F}_8$ fractions*

according to Theorem 1.1, the lattice generated by $\{O, \vec{u} = (h,k)^T, \vec{v} = (h', k')\}$ is equivalent to Λ. In other words, vectors $\vec{u}$ and $\vec{v}$ are unimodular.

Two points p and q in a lattice are said to be *mutually visible* if the segment $[pq]$ that joins them contains no other point from the lattice. In other words, the line segment $[pq]$ cannot be divided. Since each fraction $\frac{h}{k}$ in a Farey series is irreducible, it corresponds to a point (k, h) in $\mathbb{Z}^2$ mutually visible with the origin, simply called *a visible point*. The Farey series enumerates all visible points in a portion of the discrete plane $\mathbb{Z}^2$ (this area is defined by $m \geq k \geq h \geq 0$). By adding symmetric points, the Farey series $\mathcal{F}_m$ gives the whole set of visible points in a $(m+1) \times (m+1)$ square centered in the origin (Figure 1.5).

1.5. Discrete geometry elements

1.5.1. *Diophantine equations and discrete lines*

In order to define a complete digital geometrical paradigm we need points and elementary geometrical objects such as straight lines and circles. In this section, we focus on straight lines and illustrate the impact of the arithmetic results presented above on this object.

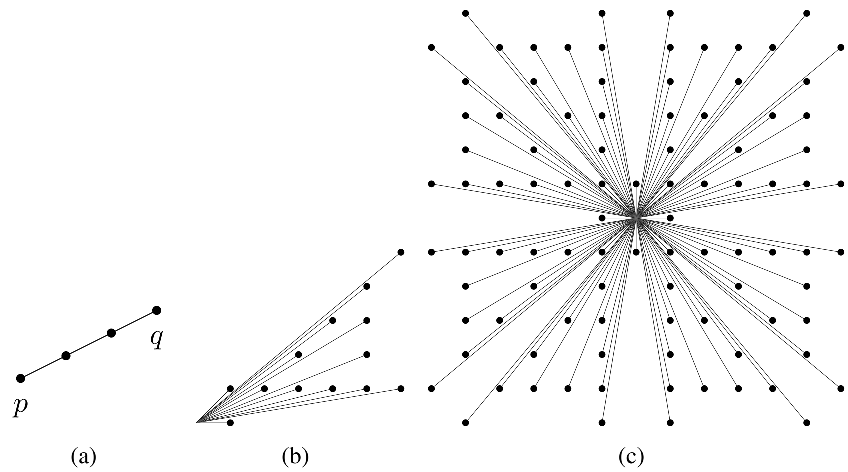

Figure 1.5. *Visible and non-visible points and Farey series: (a) points p and q are not mutually visible (the line segment $[pq]$ can be divided into three segments using lattice points between p and q); (b) graphical representation of the Farey series $\mathcal{F}_6$; and (c) the Farey series $\mathcal{F}_6$ and symmetric points produce all points visible from the origin in a centered 13×13 square*

Let us first consider the equation $ax+by = c$ which defines a straight line in the continuous case. If we consider $a, b, c \in \mathbb{Z}$ and if we are only interested in discrete solutions, then $\{(x, y) \in \mathbb{Z}^2, ax+by = c\}$ defines a *linear diophantine equation*.

First of all, if c is not divisible by $\gcd(a, b)$, no solution exists. Otherwise, the equation is equal to $\{(x, y) \in \mathbb{Z}^2, a'x + b'y = 1\}$ with $a' = \frac{a}{\gcd(a,b)}$ and $b' = \frac{b}{\gcd(a,b)}$. Using the extended Euclidean algorithm, we can compute a first solution denoted $M(m, n)$.

We can prove that the other solutions are determined by $(m - kb', n + ka')$ with $k \in \mathbb{Z}$. In other words, solutions are given by the one-dimensional (1D) lattice defined by $\vec{v} = (-b', a')$ shifted on M.

Unfortunately, this 1D lattice is not a good candidate for a discrete definition of a straight line. Basically, we would like the discrete straight line (DSL) to be a 4- or an 8-connected arc on the square grid. In the following, we consider Reveilles's analytical definition.

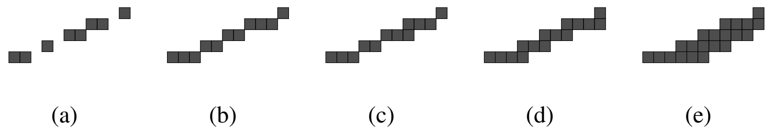

Figure 1.6. *DSL* $(3, 7, 0, \omega)$ *with variable thickness: (a)* $\omega = 5$*; (b)* $\omega = \max(|a|, |b|) = 7$*; (c)* $\omega = 8$*; (d)* $\omega = |a| + |b| = 10$*; and (e)* $\omega = 16$

DEFINITION 1.3.– *[RÉV 91] A* discrete straight line *of parameters* (a, b, μ, ω) *is the set of grid points satisfying:*

$$\mu \leq ax - by < \mu + \omega$$

with $a, b, \mu, \omega \in \mathbb{Z}$ *and* $\gcd(a, b) = 1$.

In this definition, b/a is the *slope* of the DSL, μ its intercept and ω its *thickness*. To control the topology of the discrete set, we have the following result [RÉV 91]:

THEOREM 1.2.– *Given a DSL* $D = (a, b, \mu, \omega)$*, we have:*

– *if* $\omega < \max(|a|, |b|)$*,* D *is disconnected;*

– *if* $\omega = \max(|a|, |b|)$*,* D *is an* 8*-arc;*

– *if* $\omega = |a| + |b|$*,* D *is a* 4*-arc;*

– *if* $\max(|a|, |b|) < \omega < |a| + |b|$*,* D *is an* 8*-object, but too* thin *to be a* 4*-object;*

– *if* $\omega > |a| + |b|$*,* D *is a* 4*-object, referred to as a* thick DSL.

An illustration of this theorem is given in Figure 1.6. DSL grid points defined by Definition 1.3 can also be obtained as the union of ω 1D lattices:

$$ax - by = \mu$$

$$ax - by = \mu + 1$$

$$\ldots$$

$$ax - by = \mu + \omega - 1$$

Hence, many arithmetic properties can be associated with DSL (links with digitization schemes, periodicity, etc.). For interested readers, [KLE 04] provides a good bibliography.

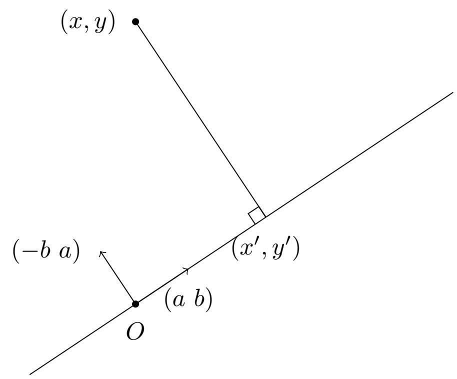

Figure 1.7. *Orthogonal projection in* $\mathbb{R}^2$

1.5.2. *Discrete projection along a rational angle*

In the remainder of the book we will mainly consider thin discrete lines with $\omega = 1$, used as discrete projection lines.

First consider the orthogonal projection in $\mathbb{R}^2$ of the point $P = (x_P, y_P)$ into $P' = (x', y')$ on the line $ax - by = 0$ as illustrated in Figure 1.7.

We have

$$\begin{cases} \begin{pmatrix} x' & y' \end{pmatrix} = t\begin{pmatrix} a & b \end{pmatrix} \\ \overrightarrow{OP'} \cdot \overrightarrow{P'P} = \begin{pmatrix} x' & y' \end{pmatrix} \cdot \begin{pmatrix} x_P - x' & y_P - y' \end{pmatrix} = 0, \end{cases}$$

$$tax_P - t^2a^2 + tbx_P - t^2b^2 = 0$$

and finally

$$\begin{pmatrix} x' & y' \end{pmatrix} = \frac{ax_P + by_P}{a^2 + b^2}\begin{pmatrix} a & b \end{pmatrix}.$$

Any point $P' = t(a, b), t \in \mathbb{R}$ on the projected line of direction vector $\begin{pmatrix} a & b \end{pmatrix}$ corresponds to an infinite set (called the *preimage* of P') of antecedent points whose projection gives P'. The preimage of P' is a thin line (a 1D lattice) whose direction vector is colinear to $\begin{pmatrix} -b & a \end{pmatrix}$ defined by: $ax + by - t/(a^2 + b^2) = 0$ or, equivalently, $ax + by - ax_P - bx_P = 0$.

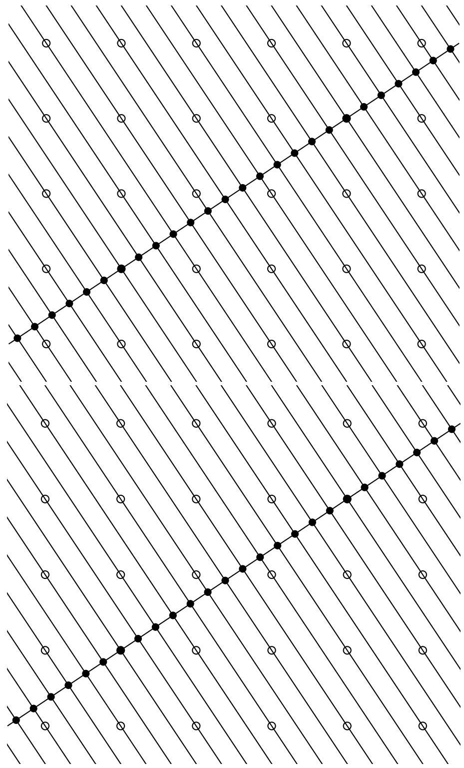

Figure 1.8. *Orthogonal projection in $\mathbb{Z}^2$ on the discrete line $-bx + ay = 0$*

If we restrict the domain (the set of source points) to $\mathbb{Z}^2$ and the direction vector $(\; a \quad b \;)$ of the projected line to $\mathbb{Z}^2 \setminus (0\;0)$, then not all points of $\mathbb{R}^2$ in the projected line $ax - by = 0$ have antecedents in $\mathbb{Z}^2$ (Figure 1.8). According to the Bézout identity, there exists $(x, y) \in \mathbb{Z}^2$ such that $ax + by = t$ if and only if $t = k \gcd(a, b), k \in \mathbb{Z}$. Any projected point (x, y) must simultaneously verify the two following equations:

$$\begin{cases} -bx + ay = 0 \\ ax + by - k \gcd(a, b) = 0. \end{cases}$$

We then have $(\; x \quad y \;) = k \gcd(a, b)(\; a \quad b \;)/(a^2 + b^2), k \in \mathbb{Z}$ and the codomain (the set of projected points) is a 1D discrete lattice whose base vector is $\frac{\gcd(a,b)(a\ b)}{a^2+b^2}$.

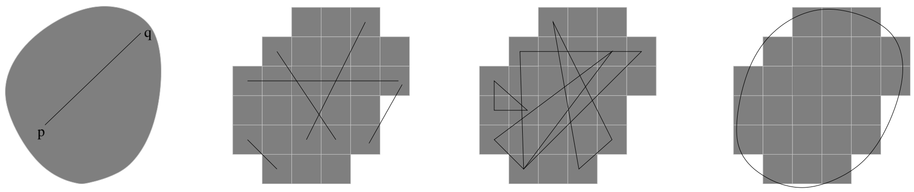

Figure 1.9. *(a) Convexity in continuous domain and illustration of discrete definitions: (b) MP-convexity; (c) V-convexity; and (d) S-convexity*

In the following, we will consider *discrete angles* in $\mathbb{Z}^2$ defined by a couple of coprime integers $(p, q) \in \mathbb{Z}^2$ with $\gcd(q, p) = 1$. We saw in section 1.4.3 how the Farey series can be used to enumerate these discrete angles of slope $q/p = 0$ to 1. For the discrete angle (p, q), we use the quantity $b = -qk + pl$ as an index to the projection of the discrete point (k, l) in the 1D lattice whose base vector is $\frac{(-q\ p)}{p^2+q^2}$.

1.5.3. *Convexity*

In the continuous domain, an object X is convex if, for any two points $p, q \in X$, the segment $[pq]$ lies in X (see Figure 1.9a). In discrete geometry, several definitions have been proposed to generalize this definition to discrete domains (Figure 1.9):

1) MP-convexity [MIN 88]: a discrete object X is *digitally convex* if, for any two point $A, B \in X$, the discrete point satisfying $[AB] \cap \mathbb{Z}^2$ is also in X.

2) V-convexity [VOS 93]: X is *digitally convex* if, for any three points $A, B, C \in X$, the discrete points in the triangle (A, B, C) lie in X.

3) S-convexity [SKL 72]: X is *digitally convex* if there exists a continuous convex object Ω such that $X = \Omega \cap \mathbb{Z}^2$.

If definitions (1) and (2) correspond to discrete versions of the continuous convexity definition, definition (3) defines the digital convexity as the result of the convex continuous object digitization. Note that this schema is usually used to define discrete geometric objects as the digitization of their continuous counterparts.

Concerning the digital convexity, even if some other definitions exist [RON 89], equivalences can be proved [KIM 80, KIM 82, ECK 01].

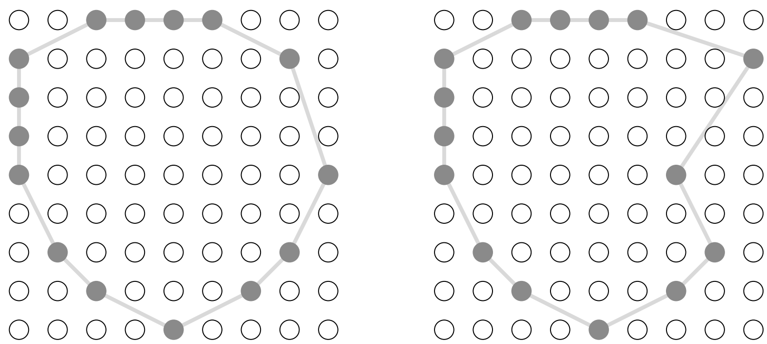

Figure 1.10. *(a, b) Lattice polygon illustrating Pick's theorem*

1.5.4. *Pick's Theorem and related counting problems*

Pick's Theorem provides a link between the area of a lattice polygon and the number of lattice points it contains. A *simple polygon* or *Jordan polygon* is a polygon whose sides do not intersect. If its vertices are lattice points, it is called a *polytgon lattice*. Let P be a lattice polygon containing I interior points and B border points. Pick's Theorem states that its area A is given by:

$$A = I + \frac{B}{2} - 1. \tag{1.2}$$

The area of the lattice polygon represented in Figure 1.10a is 48.5, the number of its interior points is 42, the number of its border points is 15 and we have $48.5 = 42 + 15/2 - 1$. In the same manner, the area of the polygon in Figure 1.10b is $45.5 = 39 + 15 + 2 - 1$.

1.5.5. *Binary mathematical morphology and two-pixel structuring elements*

Mathematical Morphology (MM), created by Georges Matheron and Jean Serra during the 1960s, is a theory dedicated to the processing of geometrical structures. It was originally developed for binary images, mainly for granulometry analysis, and was later extended to grayscale images and many spatial structures. For our purposes, we will describe the two dual operators erosion and dilation in the context of binary images. A structuring element that represents a shape is used as a probe in a binary image, to check were this shape can fit in the image. For the erosion operator, each time the translated

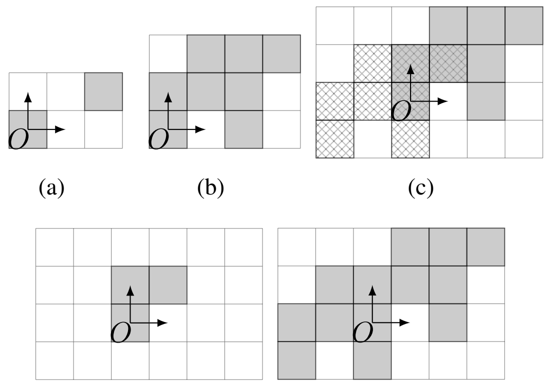

Figure 1.11. *(a) The 2PSE* $\{O, (2, 1)\}$*; (b) a binary image; (c) the image and its translation; (d) eroded image and (e) dilated image*

structuring element fits in the image, a pixel is set in the result image

$$A \ominus \check{B} = \{p \text{ s.t. } (B)_p \subseteq A\}, \tag{1.3}$$

where A is the image, B is the structuring element and $(B)_p$ is the structuring element translated by the point p.

For the dilation operator, each time the translated structuring element intersects with the image, a pixel is set in the result image

$$A \oplus \check{B} = (A^c \ominus \check{B})^c = \{p | \, (B)_p \cap A \neq \emptyset\}. \tag{1.4}$$

In the context of the Mojette transform, a specific two-pixel structuring element (2PSE) $\{O, (k, l)\}$ is defined to represent the discrete angle (k, l) (Figure 1.11). The erosion with a 2PSE is the simple intersection of the image with its translation:

$$A \ominus \check{B} = A \cap (A)_{-p}\,, \text{ where } B = \{O, p\}. \tag{1.5}$$

Compare the dilation with a 2PSE in the union of an image with its translation:

$$A \oplus \check{B} = A \cup (A)_{-p}\,, \text{ where } B = \{O, p\}. \tag{1.6}$$